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Expectation values for integer powers of a Poisson-distributed random number

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It is shown that expectation values of Poisson-distributed random numbers exist not only for the well known positive integer powers but also for negative integer powers. A recursion formula for the calculation of expectation values of powers differing by one is given. This recursion formula helps to find an analytical representation for both positive and negative integer powers in terms of the hypergeometric function.

1. Introduction

A non-negative integer random number x that is distributed according to the normalized probability density function

$$p(x) = \exp(-\lambda)\frac{\lambda^x}{x!} \tag{1}$$

with parameter λ is called a Poisson-distributed random number, or, in short, a Poisson number.

There exists a vast amount of literature about the Poisson distribution, its applications in sciences and technology and its mathematical properties. A classic textbook on this topic is by Haight (1967), which discusses the elementary properties and generalizations of the Poisson distribution, and also its applications in industry, agriculture and ecology, biology, medicine, sociology, demography and more.

The first moments about the origin are given in virtually any textbook treating the Poisson distribution. They are calculated by means of evaluating directly

$$\langle x^n \rangle = \sum_{x=0}^{\infty} x^n p(x); \quad n \in \{0, 1, 2, 3, \ldots\} \tag{2}$$

leading to

$$\langle x^1 \rangle = \lambda, \tag{3}$$

$$\langle x^2 \rangle = \lambda^2 + \lambda, \tag{4}$$

$$\langle x^3 \rangle = \lambda^3 + 3\lambda^2 + \lambda, \tag{5}$$

$$\langle x^4 \rangle = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda \tag{6}$$

and so on. Please note that a lower summation index in equation (2) starting with one instead of zero does not alter the results for $n > 0$.

The expectation values may also be calculated by using a well known recursion formula that involves differentiation with respect to λ (see *e.g.* Haight, 1967):

$$\langle x^{n+1} \rangle = \lambda\langle x^n \rangle + \lambda\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\langle x^n \rangle\right), \quad n \in \{0, 1, 2, 3, \ldots\} \tag{7}$$

with normalization $\langle x^0 \rangle = 1$. Although in the literature this recursion formula is found only for non-negative values of n, it is also valid for all negative integer values $n \leq -2$, *i.e.* all integer values of n that do not make the power of the expectation value equal to zero.

2. Another recursion formula

A different recursion formula equivalent to equation (7), but simpler, as it does not involve derivatives, is given by

$$\langle x^{n+1} \rangle = \lambda\langle (x+1)^n \rangle, \quad n \in \{\ldots, -3, -2, 0, 1, 2, \ldots\}, \tag{8}$$

i.e. any integer number n is allowed except for the one which makes the power on the left-hand side equal to zero. The derivation of this recursion formula is given in Appendix A.

3. Revision of known expectation values

With this, the expectation values [equation (3) and following] may be rewritten in the form

$$\langle x^{n+1} \rangle = \lambda \exp(-\lambda) \sum_{x=0}^{\infty} (x+1)^n \frac{\lambda^x}{x!}; \quad n \in \{0, 1, 2, 3, \ldots\}. \tag{9}$$

For example, for $n = 0$, the sum evaluates to $\exp(\lambda)$ thus leading to $\langle x \rangle = \lambda$. The sum on the right-hand side can be rewritten in terms of the hypergeometric function:

$$\langle x^{n+1} \rangle = \lambda \exp(-\lambda)\, {}_nF_{n}{}^{2,\ldots,2}_{1,\ldots,1}(\lambda); \quad n \in \{0, 1, 2, 3, \ldots\}. \tag{10}$$

Again, for $n = 0$ the hypergeometric function collapses to $\exp(\lambda)$ leading to the correct result $\langle x \rangle = \lambda$. The general form of the hypergeometric function is

$$ {}_pF_{q}{}^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}(\lambda) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{\lambda^k}{k!} \tag{11}$$

with upper coefficients a_i, $i \in \{1, 2, \ldots, p\}$ and lower coefficients b_j, $j \in \{1, 2, \ldots, q\}$. These determine the Pochhammer symbols (rising factorials)

$$(a_i)_k = \frac{\Gamma(a_i + k)}{\Gamma(a_i)}, \quad i \in \{1, \ldots p\} \tag{12}$$

and $(b_j)_k$, $j \in \{1, \ldots, q\}$. The numbers p, q and coefficients a_i and b_j have been chosen such that the term $(x+1)^n$, $n > 0$ in equation (9) is generated by the hypergeometric function in accordance with equation (8).

4. Extension to negative integer powers

Up to this point, not much has been accomplished, as the expectation values have been rewritten only in an abstract way. Equation (8), however, also holds for negative powers, $n \leq -2$, leading to expectation values of negative powers of a Poisson-distributed random number.

To see this, it is better to transform equation (8) to a formally more symmetrical appearance with the index shifted by minus one and realising that this equation holds for all positive and negative integers.

$$\langle x^n \rangle = \lambda \langle (x+1)^{n-1} \rangle, \quad n \in \{\pm 1, \pm 2, \pm 3, \ldots\} \tag{13}$$

when using the new definition

$$\langle x^n \rangle := \sum_{x=1}^{\infty} x^n p(x); \quad n \in \{\pm 1, \pm 2, \pm 3, \ldots\} \tag{14}$$

derived from equation (2) where only the lower summation index has been changed. The new definition equation (14) is identical to equation (2) in the range of validity of equation (2), but it also embraces negative integer powers.

In the case of negative integer powers, the coefficients a and b of the hypergeometric function switch roles in order to generate terms of the type $(x+1)^{-n}$, $n > 0$ in an equation similar to equation (9),

$$\langle x^{-n} \rangle = \lambda \exp(-\lambda) \sum_{x=0}^{\infty} (x+1)^{-n-1} \frac{\lambda^x}{x!}; \quad n \in \{1, 2, 3, \ldots\}, \tag{15}$$

and in accordance with equation (8), leading to

$$\langle x^{-n} \rangle = \lambda \exp(-\lambda) \; {}_{|n-1|}F_{|n-1|}{}^{1,\ldots,1}_{2,\ldots,2}(\lambda), \quad n > 0. \tag{16}$$

A potential application is in the calculation of quality indicators for data from X-ray and neutron diffraction experiments, where it is useful to calculate expectation values *e.g.* of the kind $\langle 1/I \rangle$, where I is an intensity observation of the diffracted beam (Henn & Meindl, 2010).

The following table summarizes the results for the calculation of integer powers of Poisson numbers:

$$\begin{aligned}
\langle x^n \rangle &= \lambda \exp(-\lambda) \; {}_{n-1}F_{n-1}{}^{2,\ldots,2}_{1,\ldots,1}(\lambda) \\
\vdots &= \vdots \\
\langle x^3 \rangle &= \lambda \exp(-\lambda) \; {}_2F_2{}^{2,2}_{1,1}(\lambda) \\
\langle x^2 \rangle &= \lambda \exp(-\lambda) \; {}_1F_1{}^{2}_{1}(\lambda) \\
\langle x^1 \rangle &= \lambda \exp(-\lambda) \; {}_0F_0(\lambda) \\
\langle x^{-1} \rangle &= \lambda \exp(-\lambda) \; {}_2F_2{}^{1,1}_{2,2}(\lambda) \\
\langle x^{-2} \rangle &= \lambda \exp(-\lambda) \; {}_3F_3{}^{1,1,1}_{2,2,2}(\lambda) \\
\vdots &= \vdots
\end{aligned}$$

$$\langle x^{-n} \rangle = \lambda \exp(-\lambda) \; {}_{|n-1|}F_{|n-1|}{}^{1,\ldots,1}_{2,\ldots,2}(\lambda). \tag{17}$$

The expectation value of any integer power $n \neq 0$ is expressed by the common factor $\lambda \exp(-\lambda)$ times a hypergeometric function with $p = q = |n-1|$ and $a_i \equiv 2$, $b_i \equiv 1$, $i \in \{1, \ldots, p\}$ for $n > 0$, and $a_i \equiv 1$, $b_i \equiv 2$ for $n < 0$.

5. Conclusion

Expectation values of negative integer powers of Poisson numbers exist despite possible concerns about problematic singular terms with zero in the denominator. These terms do not enter the calculation directly. The appearance of the number zero, however, is not neglected, as the full probability density function [equation (1)] is used in the calculation of the expectation values, *i.e.* the probablity density function is not renormalized to values larger than zero. This holds for positive and negative powers; they are treated in the same way. Expectation values of positive and negative integer powers of a Poisson number are expressed consistently with the help of the hypergeometric function.

APPENDIX *A*
Recursion formula

$$\begin{aligned}
\langle x^n \rangle &:= \exp(-\lambda) \sum_{x=1}^{\infty} x^n \frac{\lambda^x}{x!} \\
&= \exp(-\lambda)\lambda \sum_{x=1}^{\infty} x^{n-1} \frac{\lambda^{x-1}}{(x-1)!} \\
&= \exp(-\lambda)\lambda \sum_{y=0}^{\infty} (y+1)^{n-1} \frac{\lambda^y}{y!} \\
&= \exp(-\lambda)\lambda \sum_{x=0}^{\infty} (x+1)^{n-1} \frac{\lambda^x}{x!} \\
&= \lambda \langle (x+1)^{n-1} \rangle.
\end{aligned}$$

In line one, the expectation value of any integer power is defined such that it coincides with the usual definition for positive powers and allows for negative powers as well without generating undefined expressions with zero in the denominator. The only restriction to be made is $n \neq 0$ [this could be circumvented by adding a Kronecker delta $\delta_{n,0}$ to the sum, leading to $\langle x^n \rangle = \lambda \langle (x+1)^{n-1} \rangle + \exp(-\lambda)\delta_{n,0}$]. In line three the variable is substituted $(x-1) \rightarrow y$ and the limits are changed accordingly. In line four the variable name is changed again $y \rightarrow x$.

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